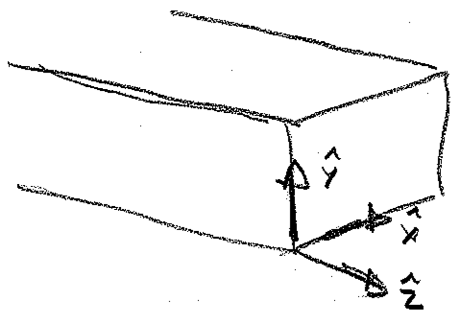
Guided Waves

$$\nabla^2 \vec{A} - \hat{y}\hat{z}\vec{A} = -\vec{J} \qquad \nabla^2 \vec{F} - \hat{y}\hat{z}\vec{F} = -\vec{M}$$

$$H = \vec{\nabla} \times \vec{A} \qquad -\vec{E} = \vec{\nabla} \times \vec{F}$$

$$\vec{E} = -\hat{z}\vec{A} + \frac{1}{\hat{y}}\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) \qquad \vec{H} = -\hat{y}\vec{F} + \frac{1}{\hat{z}}\vec{\nabla}(\vec{\nabla}\cdot\vec{F})$$

Consider a rectangular waveguide with coordinate system



A Transverse Electric & Magnetic field mode (TEM) needs a conductor for the magnetic field to circulate around.

A wave guide does not provide this.

A TEM mode propagets all the way down to DC.

i.e. A waveguide mode will not work at DC. = OR. = The waveguide walls <u>short out</u> the DC mode

Now Consider Transverse Electric to z (TE_z) or Transverse magnetic to z (TM_z)

To get transverse electric to z we need to consider.

$$\vec{E} = \vec{\nabla} \times \vec{F}$$

If $\qquad \vec{F} = \psi(x,y,z)\,\hat{z}$

then $\qquad E_z = 0$

$$\nabla^2 \vec{F} - \hat{y}\hat{z}\ \vec{F} = \vec{M}$$

Becomes

$$\nabla^2 \psi + \omega^2 \mu \varepsilon\ \psi = 0$$

Let $\qquad k^2 = \omega^2 \mu \varepsilon$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0$$

Try separating the Variables with

$$\psi = X(x)\ Y(y)\ Z(z)$$

$$YZ \frac{d^2 X}{dx^2} + XZ \frac{d^2 Y}{dy^2} + XY \frac{\partial^2 Z}{\partial z^2} + k^2 XYZ = 0$$

Divide Both sides by XYZ

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} + k^2 = 0$$

This can only work if

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^2$$

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^2$$

$$\frac{1}{Z}\frac{\partial^2 Y}{\partial z^2} = -k_z^2$$

& $\quad k_x^2 + k_y^2 + k_z^2 = k^2$

This is called the separation Equation.

$$\frac{\partial^2 X}{\partial x^2} + k_x^2 X = 0$$

$$\frac{\partial^2 Y}{\partial y^2} + k_y^2 Y = 0$$

$$\frac{\partial^2 Z}{\partial z^2} + k_z^2 Z = 0$$

$$X = \alpha_x e^{-jk_x x} + \beta_x e^{jk_x x}$$

$$Y = \alpha_y e^{-jk_y y} + \beta_y e^{jk_y y}$$

$$Z = \alpha_z e^{-jk_z z} + \beta_z e^{jk_z z}$$

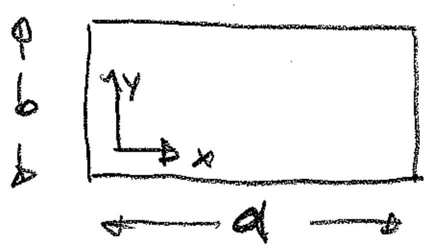
Look at the electric field

$$E = -\nabla \times F$$

$$= -\nabla \times \psi \hat{z} = -\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \psi \end{vmatrix}$$

$$E_x = -\frac{\partial \psi}{\partial y} = -X \frac{\partial Y}{\partial y} Z$$

$$E_y = \frac{\partial \psi}{\partial x} = \frac{\partial X}{\partial x} Y Z$$



Boundary conditions $E_x = 0$ at $y = 0, b$

$$\frac{\partial Y}{\partial y} \propto \sin\left(\frac{n\pi y}{b}\right)$$

$$Y \propto \cos\left(\frac{n\pi y}{b}\right) \qquad n = 0, 1, 2, 3$$

$$k_y = \frac{n\pi}{b}$$

$E_y = 0$ at $x = 0, a$

$$\frac{\partial X}{\partial x} \propto \sin\left(\frac{m\pi x}{a}\right)$$

$$X \propto \cos\left(\frac{m\pi x}{a}\right)$$

$$k_x = \frac{m\pi}{a}$$

For the z direction we want propagating waves; Not standing waves

$$Z(z) = e^{-jk_z z}$$

is a forward propagating wave

$$Z(z) = e^{jk_z z}$$

is a reverse propagating wave

$$\psi^{\pm} = F_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) e^{\mp jk_z z}$$

where

$$\omega^2\mu\varepsilon = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + k_z^2$$

$$k_z^2 = \omega^2 \mu \varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2$$

for $m, n = 0, 1, 2 \ldots$

What happens if m, n is too big?

$$k_z^2 < 0$$

$$k_z = \mp j |k_z|$$

$$Z^{\pm}(z) = e^{\mp j(\mp j |k_z| z)}$$

$$= e^{\mp |k_z| z}$$

This is not a wave but a damped exponential. This wave is sayed to be cut off

The cut off frequency occurs when

$$k_z = 0$$

$$f_{c_{m,n}}^2 = \left(\frac{c}{2\pi}\right)^2 \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]$$

or

$$f_{c_{m,n}} = c^2 \left[\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2\right]$$

Can m & n be zero at the same time ?

$$F^{+} = \hat{z} F_{oz} \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) e^{-jk_z z}$$

$$E_x = + F_{oz}\left(\frac{n\pi}{b}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) e^{-jk_z z}$$

$$E_y = - F_{oz}\left(\frac{m\pi}{a}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) e^{-jk_z z}$$

So if m & n = 0 at the same time then $E_x = E_y = 0$.

Boring!!

What is the lowest frequency that can propagate ?

Assume $a > b$

Therefore lowest frequency occurs for

$$m = 1 \qquad n = 0$$

TE_{10} mode

$$f_c = \frac{c}{2a}$$

Or when the width becomes $\frac{\lambda}{2}$

Look at the magnetic field

$$-\frac{1}{j\omega\mu} \nabla \times E = H$$

$$\vec{H} = \frac{-1}{j\omega\mu} \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & 0 \end{vmatrix}$$

$$H_x = \frac{+1}{j\omega\mu} \frac{\partial E_y}{\partial z} = \frac{-k_z}{\omega\mu} E_y = \frac{-k_z}{\omega\mu} \frac{\partial \psi}{\partial x}$$

$$H_y = \frac{-1}{j\omega\mu} \frac{\partial E_x}{\partial z} = \frac{k_z}{\omega\mu} E_x = \frac{+k_z}{\omega\mu} \frac{\partial \psi}{\partial y}$$

$$H_z = -\frac{1}{j\omega\mu}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right)$$

$$= \frac{-1}{j\omega\mu}\left(\frac{\partial^2}{\partial x^2}\psi + \frac{\partial^2}{\partial y^2}\psi \right)$$

E | H

$$E_x = -\frac{\partial \psi}{\partial y} \qquad H_x = -\frac{k_z}{\omega\mu}\frac{\partial \psi}{\partial x}$$

$$E_y = \frac{\partial \psi}{\partial x} \qquad H_y = -\frac{k_z}{\omega\mu}\frac{\partial \psi}{\partial y}$$

$$E_z = 0 \qquad H_z = \frac{-1}{j\omega\mu}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right)$$

$$\psi = F_{oz}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-jk_z z}$$

$$k_z^2 = \omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 = \omega^2\mu\varepsilon - \omega_c^2\mu\varepsilon$$

$$E_x = +F_{oz}\left(\frac{n\pi}{b}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-jk_z z}$$

$$E_y = -F_{oz}\left(\frac{m\pi}{a}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-jk_z z}$$

$$E_z = 0$$

$$H_x = +F_{oz}\left(\frac{k_z}{\omega\mu}\right)\left(\frac{m\pi}{a}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-jk_z z}$$

$$H_y = +F_{oz}\left(\frac{k_z}{\omega\mu}\right)\left(\frac{n\pi}{b}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-jk_z z}$$

$$H_z = \frac{-j}{\omega\mu}F_{oz}\,\omega_c^2\mu\varepsilon\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-jk_z z}$$

$$H_z = -j\left(\frac{\omega_c}{\omega}\right)\omega_c \varepsilon F_{oz} \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) e^{-jkz}$$

Look at the TE_{10} mode

$$E_x = 0$$

$$E_y = -F_{oz}\left(\frac{\pi}{a}\right) \sin\left(\frac{m\pi x}{a}\right) e^{-jk_c z}$$

$$E_z = 0$$

$$H_x = F_{oz}\left(\frac{k_z}{\omega\mu}\right)\left(\frac{\pi}{a}\right) \sin\left(\frac{m\pi x}{a}\right) e^{-jk_c z}$$

$$H_y = 0$$

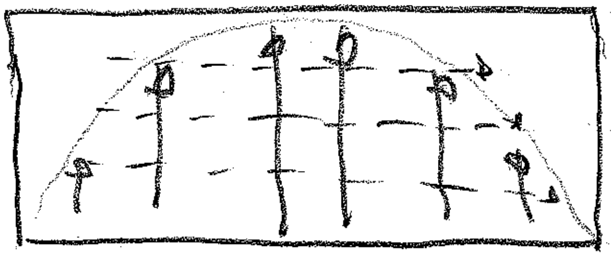
$$H_z = -j\omega_c \varepsilon \left(\frac{\omega_c}{\omega}\right) F_{oz} \cos\left(\frac{m\pi x}{a}\right) e^{-jk_c z}$$

As $\omega \to \infty$ $H_z \to 0$

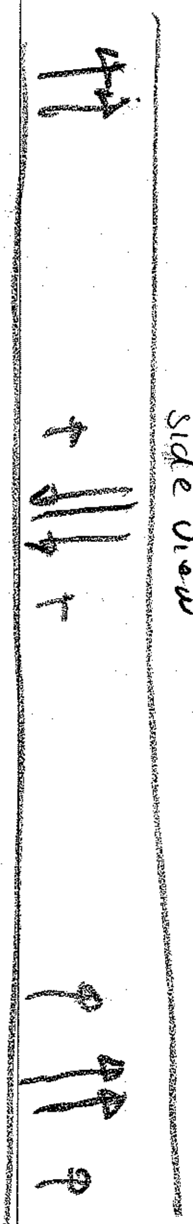
$\therefore$ TEM mode

Same as shining a flash light down a pipe.

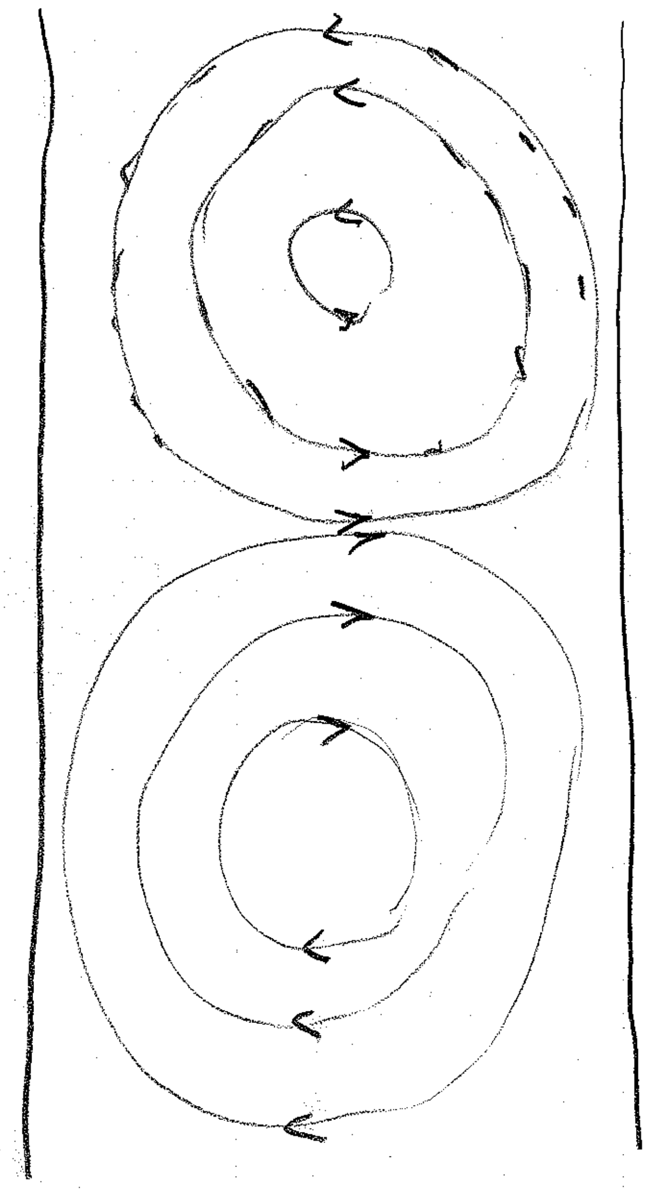
TE_{10} Mode



Back view



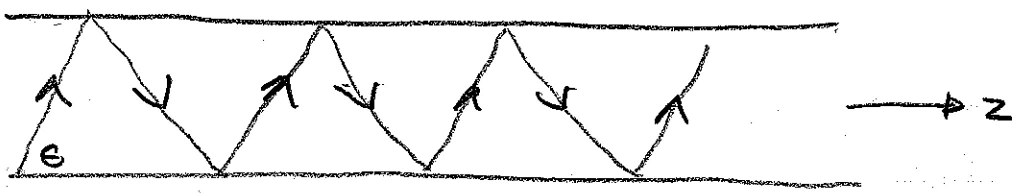
Side view

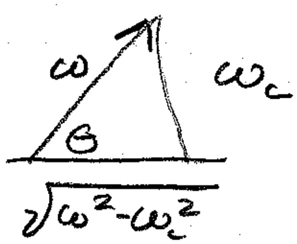


Top view

What does cut-off really mean?

Think of a waveguide as containing a TEM mode that bounces off the walls





as $\omega \to \omega_c$ $\quad \theta \Rightarrow 90^\circ$

Wave doesn't propagate.

Phase Velocity

$$E_y = -F_{0z}\left(\frac{\pi}{a}\right)\sin\left(\frac{m\pi x}{a}\right) e^{j\omega t} e^{-jk_z z}$$

$$E_y = -F_{0z}\left(\frac{\pi}{a}\right)\sin\left(\frac{m\pi x}{a}\right) e^{jk_z\left(\frac{\omega}{k_z}t - z\right)}$$

The phase of the wave is

$$\Theta = k_z\left(\frac{\omega t}{k_z} - z\right)$$

The "velocity" of the wave front is called the phase velocity

$$v_{ph} = \frac{\omega}{k_z}$$

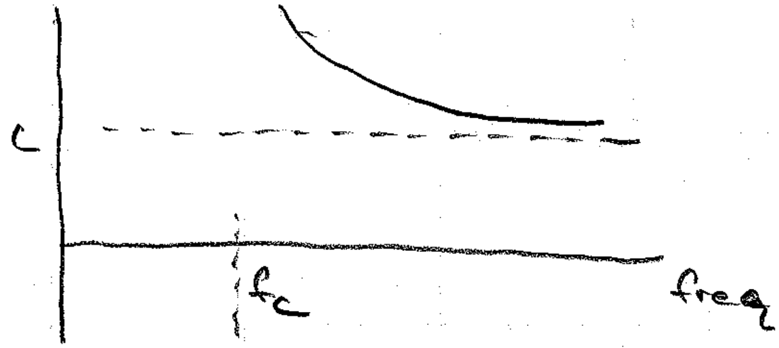
$$k_z^2 = \frac{1}{c^2}\left(\omega^2 - \omega_c^2\right)$$

$$v_{ph} = \frac{c}{\sqrt{1 - \frac{\omega_c^2}{\omega}}}$$

For $\omega < \omega_c$

v_{ph} is imaginary

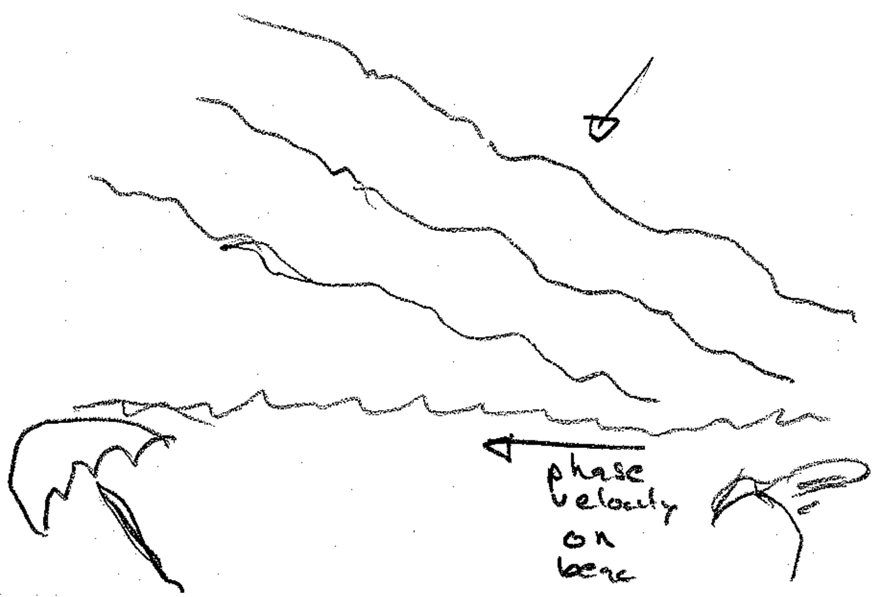
wave doesn't propagate



Just above f_c, the phase velocity $>> c$!!!

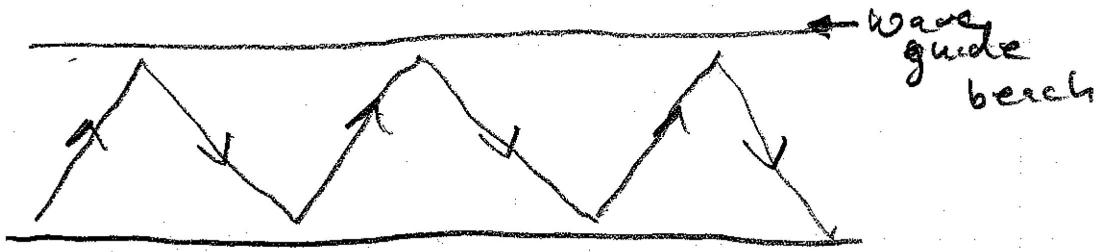
How is this possible?

Consider waves on a beach



As the waves hit the beach the wave front as it hits the sand "rips" along the beach.

Go back to the bouncing waveguide



What is physical about the phase velocity?

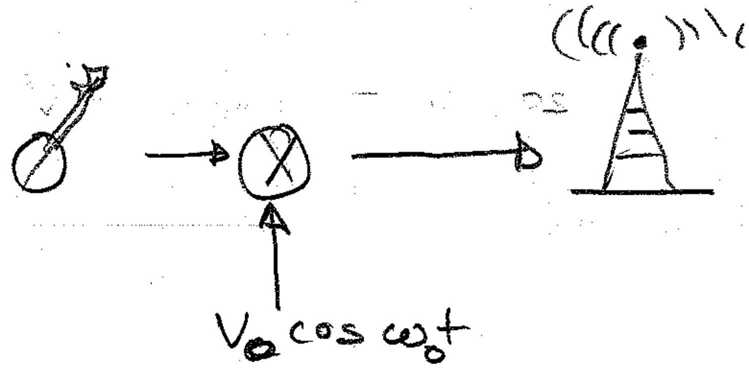
* A pure sine wave has no information content (kind of like FOX News)

To send information there must be some modulation of the sine wave

Consider some banjo music

$$V_{Banjo} = m V_0 \cos(\Delta\omega t)$$

Send this out an a transmitter



$$V = V_0 \left(1 + m\cos(\Delta\omega t)\right)\cos\omega_0 t$$

$$V = V_0 \cos\omega_0 t + V_0 \frac{m}{2}\left[\cos((\omega_0+\Delta\omega)t + \cos((\omega_0-\Delta\omega)t\right]$$

As the waves emanate from the source

$$V = V_0 \cos(\omega_0 t - \beta_0 z)$$

$$+ V_0 \frac{m}{2} \cos\Big((\omega_0 + \Delta\omega)t - (\beta_0 + \Delta\beta) z\Big)$$

$$+ V_0 \frac{m}{2} \cos\Big((\omega_0 - \Delta\omega)t - (\beta_0 - \Delta\beta) z\Big)$$

Which can be re-written as

$$V = V_0 \Big(1 + m\cos(\Delta\omega t - \Delta\beta z)\Big) \cos(\omega t - \beta z)$$

The information travels at

$$v_{info} = \frac{1}{\frac{\Delta\beta}{\Delta\omega}}$$

$$\Rightarrow \qquad v_{info} = \frac{1}{\frac{\partial\beta}{\partial\omega}}$$

For a waveguide

$$k_z^2 = \frac{1}{c^2}(\omega^2 - \omega_c^2)$$

$$2k_z \frac{\partial k_z}{\partial\omega} = \frac{2\omega}{c^2}$$

$$\frac{\partial k_z}{\partial \omega} = \frac{\omega / c^2}{\frac{1}{c}\sqrt{\omega^2-\omega_c^2}}$$

$$= \frac{1/c}{\sqrt{1-\frac{\omega_c^2}{\omega^2}}}$$

$$v_{info} = c\sqrt{1-\frac{\omega_c^2}{\omega^2}}$$