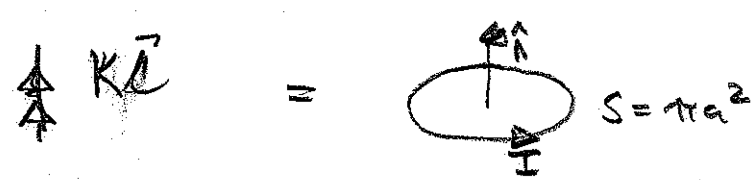
Duality

Electric	Magnetic
$\vec{\nabla} \times \vec{H} = \hat{y}\,\vec{E} + \vec{J}$	$-\nabla \times \vec{E} = \hat{z}\,\vec{H} + \vec{M}$
$-\nabla \times \vec{E} = \hat{z}\,\vec{H}$	$\nabla \times \vec{H} = \hat{y}\,E$
$\vec{H} = \vec{\nabla} \times \vec{A}$	$\vec{E} = -\vec{\nabla} \times \vec{F}$
$\vec{E} = -\hat{z}\vec{A} + \frac{1}{\hat{y}}\vec{\nabla}(\vec{\nabla}\cdot\vec{A})$	$\vec{H} = -\hat{y}\vec{F} + \frac{1}{\hat{z}}\vec{\nabla}(\vec{\nabla}\cdot\vec{F})$
$\nabla^2\vec{A} - \hat{y}\hat{z}\,\vec{A} = -\vec{J}$	$\nabla^2\vec{F} - \hat{y}\hat{z}\,\vec{F} = -\vec{M}$
$A(\vec{r}) = \iiint J(\vec{r}\,')\dfrac{e^{-jk\lvert\vec{r}-\vec{r}\,'\rvert}}{\lvert\vec{r}-\vec{r}\,'\rvert}\,dv$	$F(\vec{r}) = \iiint M(\vec{r}\,')\dfrac{e^{-jk\lvert\vec{r}-\vec{r}\,'\rvert}}{\lvert\vec{r}-\vec{r}\,'\rvert}\,dv$

What is a magnetic current? (Again)

One can show using equations above that fields from the below sources are equal



If

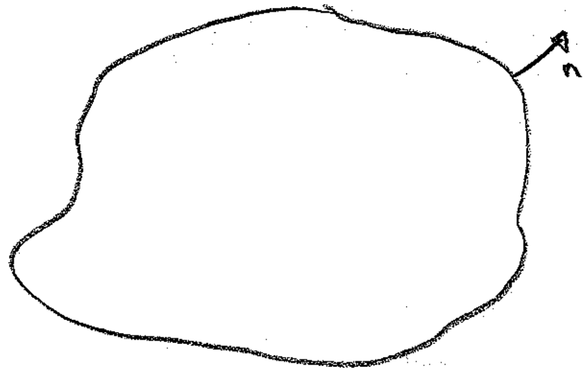
$$K\vec{l} = \hat{z}\, I S\, \hat{n}$$

units of Volts

units of Ω/m

Uniqueness

Consider a surface



Now consider two solutions

$$\vec{E}^a, \vec{H}^a \quad \text{an} \quad \vec{E}^b, \vec{H}^b$$

Define the difference solution

$$\delta\vec{E} = \vec{E}^a - \vec{E}^b$$

$$\delta\vec{H} = \vec{H}^a - \vec{H}^b$$

$$\left.\begin{aligned} -\nabla \times \delta\vec{E} &= \hat{z}\,\delta\vec{H} \\ \nabla \times \delta\vec{H} &= \hat{y}\,\delta\vec{E} \end{aligned}\right\} \text{within } S$$

From Power Derivations

$$\oiint (\delta E \times \delta H^*) \cdot dS$$

$$= \iiint \left(\hat{z}\,|\delta\vec{H}|^2 + \hat{y}^*\,|\delta E|^2\right) dv$$

Let us <u>force</u> $\delta\vec{E} = 0$ & $\delta\vec{H} = 0$ <u>on</u> S

$$\therefore \oiint (\delta\vec{E} \times \delta\vec{H}^*) \cdot ds = 0$$

So $\iiint \left(Re(\hat{z}) |\delta H|^2 + Re(\hat{y}) |\delta E|^2 \right) dv = 0$

& $\iiint \left(Im(\hat{z}) |\delta H|^2 - Im(\hat{y}) |\delta E|^2 \right) dv = 0$

For a dissapative medium

$$Re(\hat{z}) > 0$$
$$Re(\hat{y}) > 0$$

$\therefore$ $\delta\vec{H} = 0$ & $\delta\vec{E} = 0$ <u>IN</u> S

Summary: If we specify the fields <u>on</u> S

There is only one unique solution <u>in</u> S

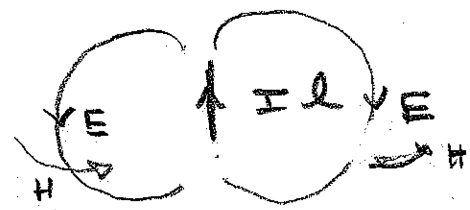
Actually its even more restrictive
There is a unique solution <u>IN</u> S
If 1) $n \times \delta\vec{E} = 0$ <u>on</u> S

or 2) $n \times \delta\vec{H} = 0$ <u>on</u> S

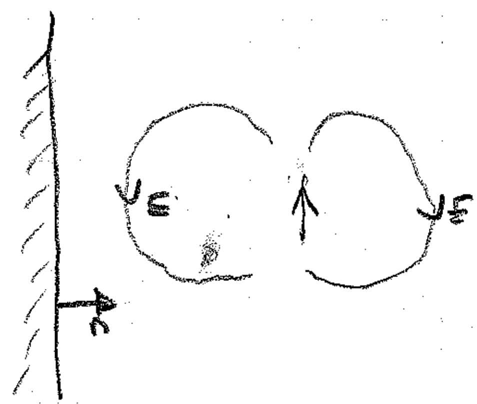
or 3) $n \times \delta E = 0$ on part of S & $n \times \delta H = 0$ other S.

Image Theory

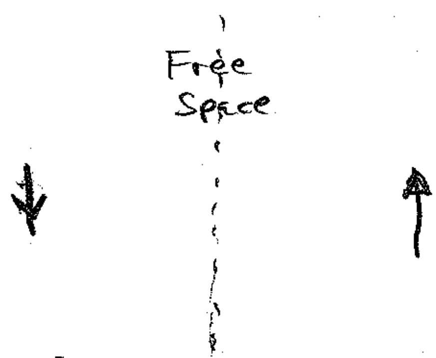
Consider a short electric dipole



Put the dipole next to the ground plane

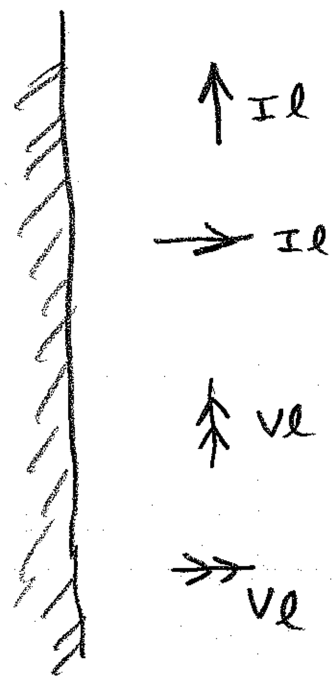


We must have $\vec{n} \times \vec{E} = 0$ on the ground plane

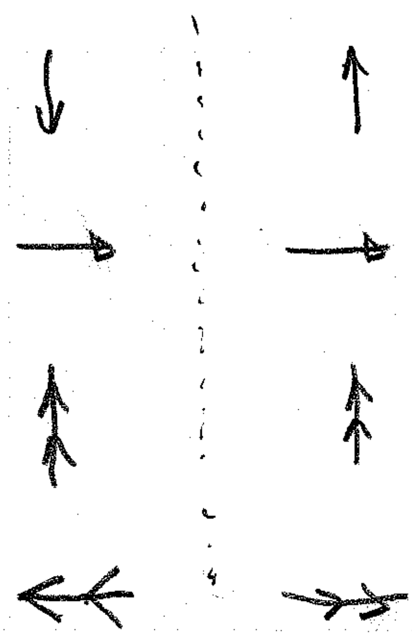


Satisfies this requirement. & From Uniqueness Theorem is the only solution

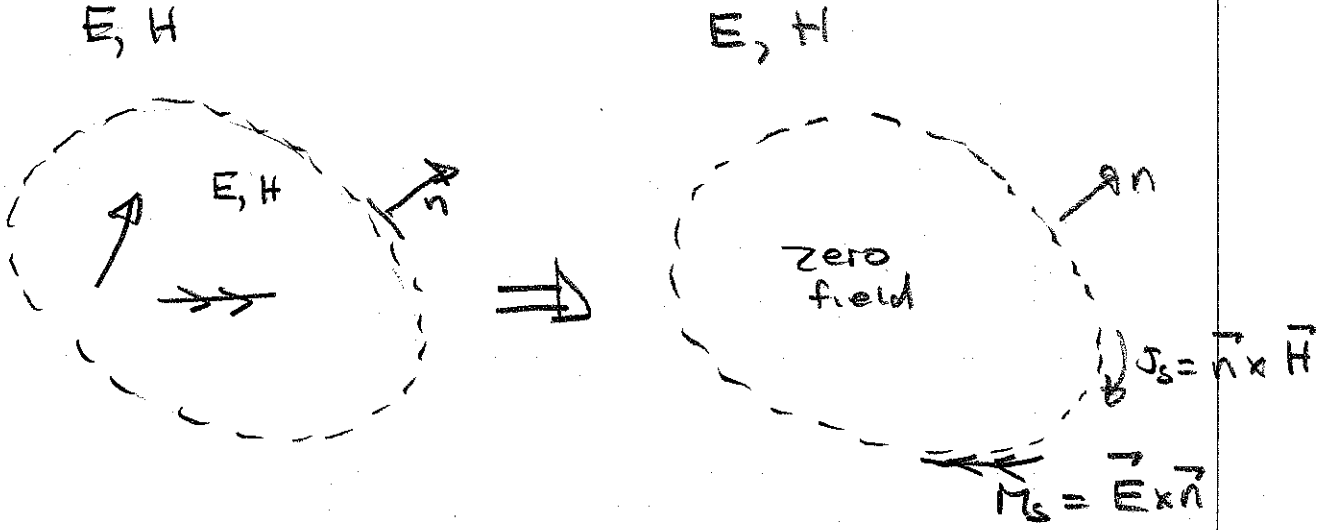
Like Wise



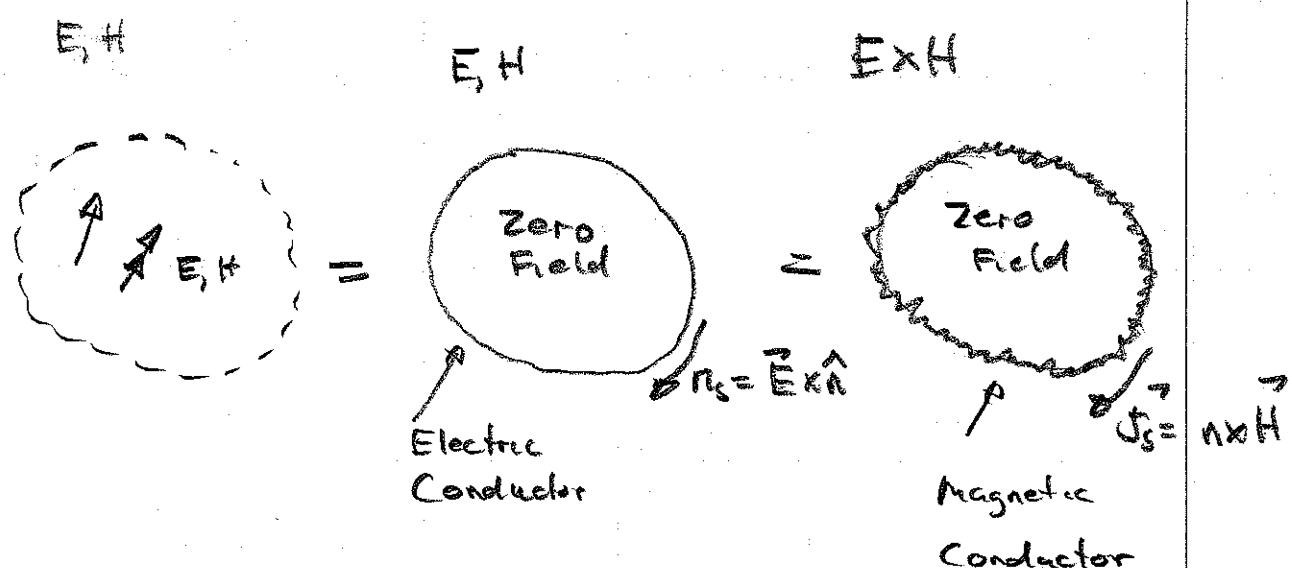
Is the same as:



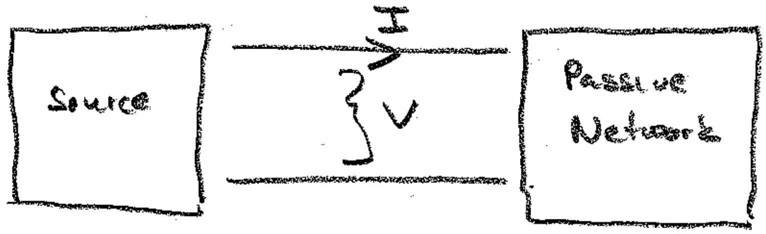
Equivalence Principle

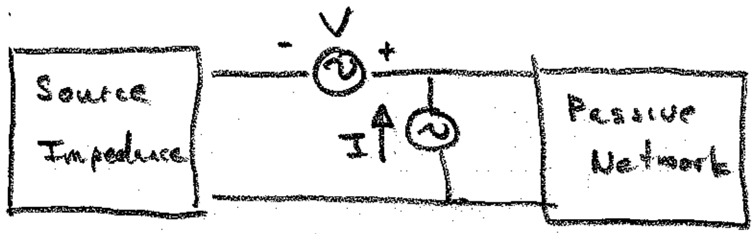


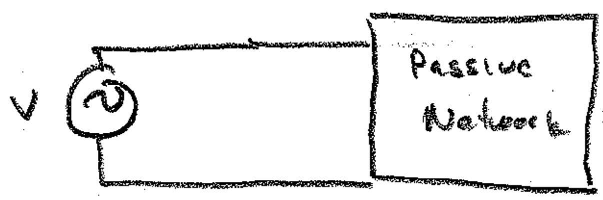
Since the field is zero inside S we can put any surface we want

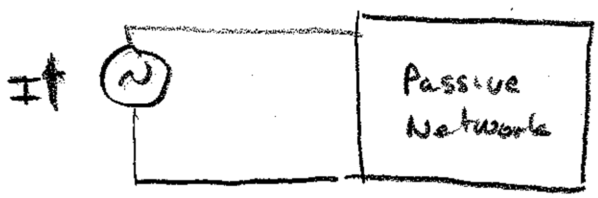


Equivalence Principle in Circuit Theory

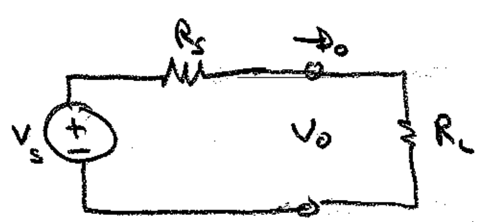








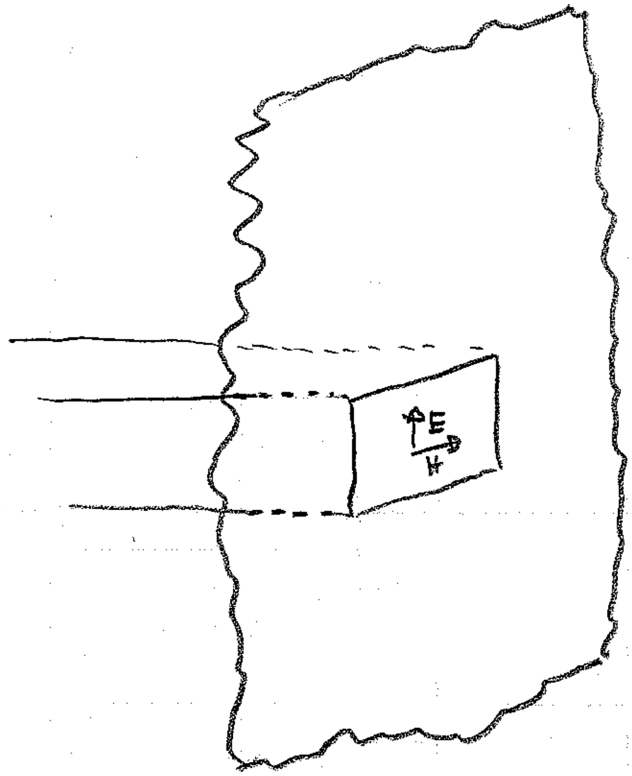
Check

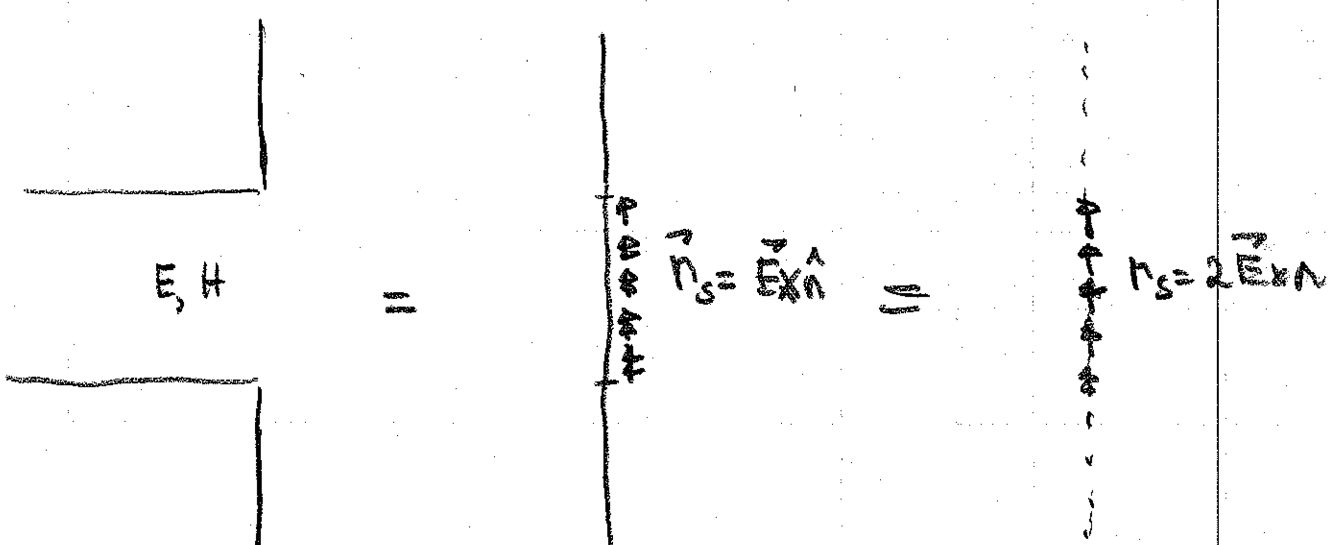


$$V_o = \frac{R_L}{R_S + R_L} V_S$$

$$I_o = \frac{V_s}{R_L + R_S}$$

Fields in a Half Space





$$\vec{F}(\vec{r}) = \iint_S 2\vec{E}(r') \times \hat{n} \, \frac{e^{-jk|r-r'|}}{|\vec{r}-\vec{r}'|} \, d$$

Note that this is an Integral Equation

$$E = \nabla \times F(r)$$

$$E(r) = \nabla \times \iint_S 2\, E(r') \times \hat{n} \, \frac{e^{-ik|r-r'|}}{|r-r'|} \, ds'$$

How does one solve integral equations

☆ 1) Guess at $E(r')$ on the surface

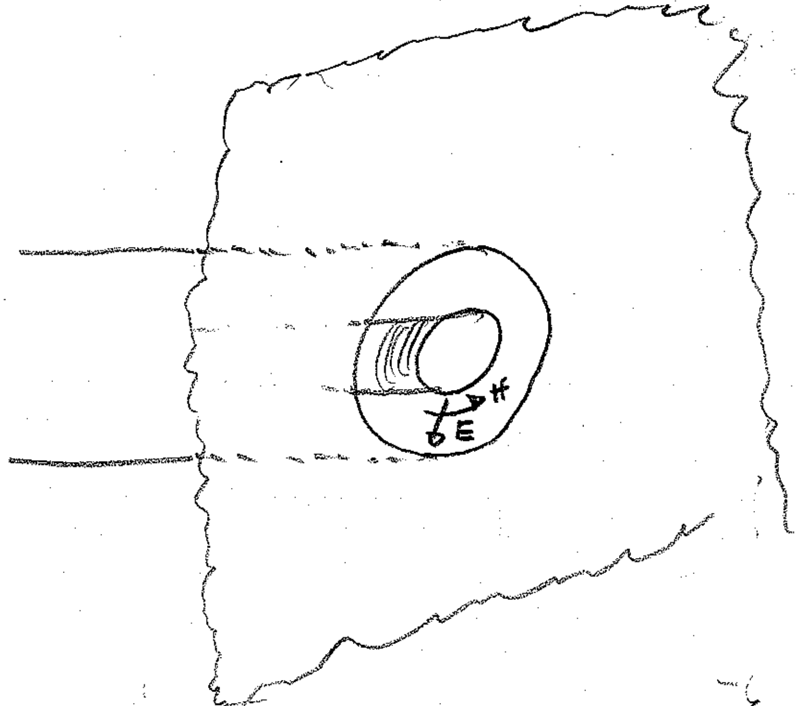
usually ok

2) a) Guess at $E(r')$ on surface

b) come up with error term and use an apposcimation for soluing error term et.

3) On the surface, expand $E(r)$ in terms of basis functions. Then multiply integral equation by basis functions (Galerkin approach) & integrate. The integral equation becomes a matrix equation. This is the basis for moment methods.

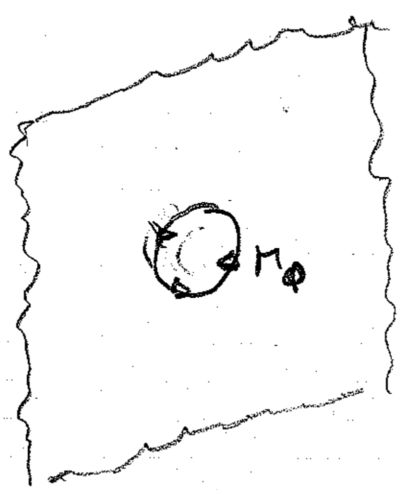
Coaxial line opening into a half space



Note that the current at the end of an open coax is zero.

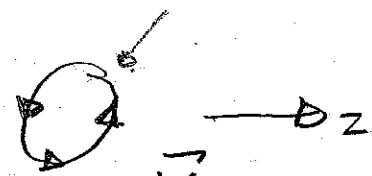
Assume that the voltage at the end of the coax is V_0

$$E_\rho = \frac{-V_0}{\rho \ln(b/a)}$$ (from statics TEM mode)



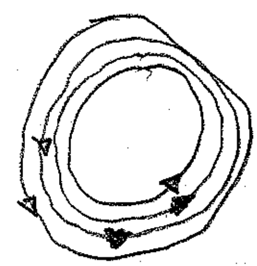
where $\vec{M}_\phi = E_\rho \hat{\phi}$

We can use method of images



$$\vec{M}_{\phi_f} = 2E_\rho\,\hat{\phi}$$

Since the E field is spread out over the radius & if $\underline{\underline{\underline{b \ll \lambda}}}$

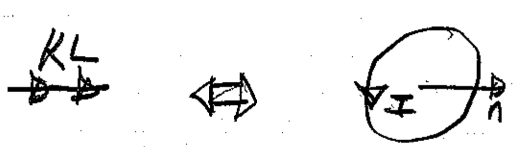


Magnetic Current Loop $\Delta KS = (\pi\rho^2)\,2E_\rho(\rho)\,\Delta\rho$

$$\langle KS \rangle = \int_a^b \pi\rho^2\,2E_\rho(\rho)\,d\rho$$

$$= \pi V \frac{(b^2-a^2)}{\ln(b/a)}$$

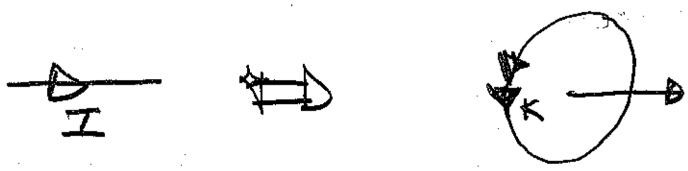
Remember



$$Kl = \hat{z}\,I\,S\,\hat{n}$$

$$\hat{z} = j\omega\mu$$

The Dual is also correct



$$\text{if } I\vec{l} = j\omega\varepsilon K S \hat{n}$$

$$I\vec{l} = \hat{y} K S \hat{n}$$

units of A

units of $\frac{1}{\Omega m}$

units of V

$$I\vec{l} \cong \hat{y} \frac{V(b^2-a^2)}{\ln(b/a)}$$

$$\vec{A} = \hat{y} \frac{V(b^2-a^2)}{\ln(b/a)} \vec{Z} \frac{e^{-jk|r|}}{|r|}$$

$$\vec{H} = \vec{\nabla} \times \vec{A}$$

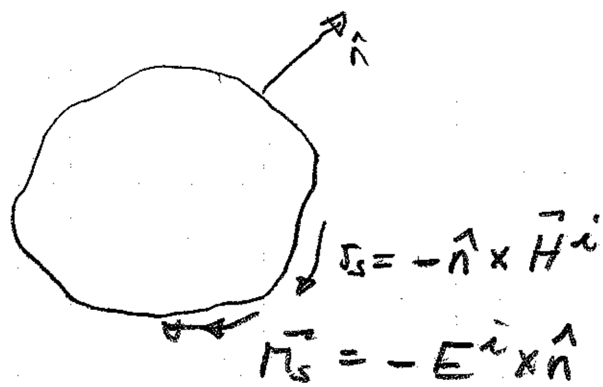
$$\vec{E} = -\vec{z}\vec{A} + \frac{1}{\hat{y}} \vec{\nabla}(\vec{\nabla}\cdot\vec{A})$$

Induction Theorem

$$E = E^i + E^s$$

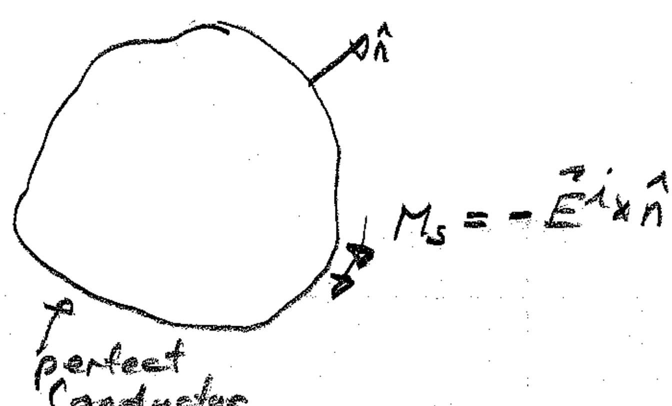
If we do not need to know the field inside the obstacle

$$\vec{E}_s$$



What if the obstacle is a perfect conductor

$$\vec{E}_s$$



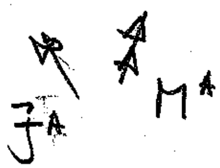
J_s is shorted out on a perfect conductor.

Reciprocity

Consider 2 problems

Problem A

$\vec{E}^a, \vec{H}^a$



$$\vec{\nabla}\times\vec{H}^a = y\vec{E}^a + \vec{J}^a$$
$$-\vec{\nabla}\times\vec{E}^a = z\vec{H}^a + \vec{M}^a$$

Problem B

$\vec{E}^b, \vec{H}^b$

$$\nabla\times H^b = yE^b + J^b$$
$$-\nabla\times E^b = zH^b + M^b$$

Using:

$$\nabla\cdot(A\times B) = B\cdot\nabla\times A - A\cdot\nabla\times B$$

Derive

$$-\nabla\cdot(E^a \times H^b - E^b \times H^a) = E^a\cdot J^b - E^b\cdot J^a - (H^a\cdot J^b - H^b\cdot M^a)$$

Integrate over Space

$$\oiint((E^a \times H^b)-(E^b \times H^a))\cdot dS$$
$$=\iiint(E^a\cdot J^b - H^a\cdot M^b)\,dV - \iiint(E^b\cdot J^a - H^b\cdot M^a)\,dV$$

Bring the surface out to infinity

$$\oiint((E^a \times H^b)-(E^b \times H^a))\cdot dS = 0$$

because fields die away.

☆ $$\iiint(E^a\cdot J^b - H^a\cdot M^b)\,dV = \iiint(E^b\cdot J^a - H^b\cdot M^a)\,dV$$

What does this mean?

Consider $\vec{J}^b = \vec{I}^b\,\delta(x-x_b')\delta(y-y_b')$ $\quad M^b = 0$

$\vec{J}^a = I^a\,\delta(x-x_a')\delta(y-y_a')$ $\quad M^a = 0$

$$I^b\int E^a\cdot dl_b = I^a\int E^b\cdot dl_a$$

$$I^b\,V^{ba} = I^a\,V^{ab}$$

V^{ba}: voltage at b due to source at a

V^{ab}: voltage at a due to a source at b

$$\begin{bmatrix} V_a \\ V_b \end{bmatrix} = \begin{bmatrix} Z_{AA} & Z_{AB} \\ Z_{BA} & Z_{BB} \end{bmatrix} \begin{bmatrix} I_a \\ I_b \end{bmatrix}$$

Reciprocity means that

$$Z_{AB} = Z_{BA}$$

OR: The transmitting pattern is the same as the recieving pattern!

Physical Optics Approximation

Consider a region that is mostly emply but has an obstagle

The material properties of the region is a function of position

$$\hat{z} = \hat{z}_T(\vec{r},\omega) \qquad \hat{y} = y_T(\vec{r},\omega)$$

$$-\nabla\times\vec{E} = z_T(\hat{r})\vec{H} + \vec{M}^i \qquad \nabla\times\vec{H} = \hat{y}_T(\vec{r})E + J^i$$

Outside S the material properties are constant

$$\hat{z} = \hat{z}_0(\omega) \qquad \hat{y} = y_0(\omega)$$

We can re-write the problem as

$$-\vec{\nabla}\times\vec{E} = \hat{z}_0\vec{H} + (\hat{z}_T - \hat{z}_0)\vec{H} + \vec{M}^i$$

$$\vec{\nabla}\times\vec{H} = \hat{y}_0\vec{E} + (\hat{y}_T - \hat{y}_0)\vec{E} + \vec{J}^i$$

Decompose the total field into incident and scattered fields

$$\vec{E}^T = \vec{E}^i + \vec{E}^S$$

$$\vec{H}^T = \vec{H}^T + \vec{H}^s$$

$$-\vec{\nabla}\times(\vec{E}^i + \vec{E}^S) = \hat{Z}_0(\vec{H}^i + \vec{H}^s) + (\hat{Z}_T - \hat{Z}_0)(\vec{H}^i + \vec{H}^s) + \vec{M}^i$$

$$\vec{\nabla}\times(\vec{H}^i + H^s) = Y_0(\vec{E}^i + \vec{E}^S) + (Y_T - Y_0)(\vec{E}^i + \vec{E}^S) + \vec{J}^i$$

Incident problem

$$-\vec{\nabla}\times\vec{E}^i = \hat{Z}_0 \vec{H}^i + \vec{M}^i$$

$$-\vec{\nabla}\times\vec{H}^i = \hat{Y}_0 \vec{E}^s + \vec{J}^i$$

Scattered problem

$$-\vec{\nabla}\times\vec{E}^s = \hat{Z}_0 \vec{E}^s + \vec{M}^s$$

$$\vec{\nabla}\times\vec{H}^s = \hat{Y}_0 \vec{E}^s + \vec{J}^s$$

free space

$$\vec{M}^s = (\hat{Z}_T - \hat{Z}_0) H^T = j\omega(\mu_T - \mu_0) H^T$$

free space

$$\vec{J}^s = (\hat{Y}_T - \hat{Y}_0) E^T = j\omega(\varepsilon_T - \varepsilon_0) E^T + \sigma_T E^T$$

Assume a non-magnetic <u>metal</u> object

since $\mu_r = \mu_0$ $\quad M^s = 0$ $\quad \vec{F}^s = 0$

$\varepsilon_T = \varepsilon_0$ $\quad J^s = \sigma_T E^T$

$$\vec{A}_S = \frac{1}{4\pi} \iiint_{\text{Volume}} \sigma_T \vec{E}^T_T \frac{e^{-ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} dV'$$

as $\sigma \Rightarrow \infty$ $\quad \sigma_T E^T$ becomes a <u>true</u> surface current

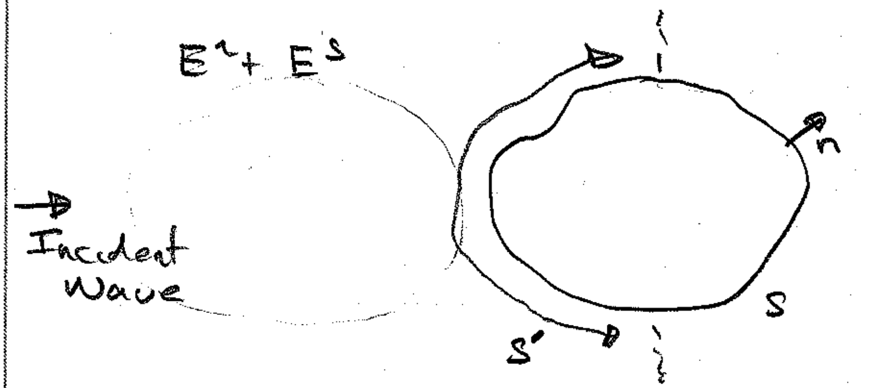
$$\vec{A}_S = \frac{1}{4\pi} \oiint_{\text{Surface}} J_{\text{surface}}(E^T) \frac{e^{-ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} dS'$$

The surface current on a conductor is given by

$$J_{\text{surface}} = \hat{n} \times \vec{H}_{\text{surface}}$$

<u>Physical Optics Approx</u>

1) Assume object large
2) Field negligible in shadow region
3) Surface of object "smooth"



Using image theory we can assume

$$\vec{J}_{\substack{surface \\ on\ S'}} \approx 2\vec{n} \times \vec{H}_i$$

$$\vec{A}_s = \frac{1}{4\pi} \iint_{S'} 2\hat{n} \times \vec{H}_i(\bar{r}') \frac{e^{-ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} \, ds'$$